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Is Weibull's modulus really a material constant? Example case with interacting collinear cracks

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Abstract

The Weibull distribution is widely used to describe the scatter of the strength in brittle (but also quasi-brittle) materials, often assuming that the Weibull modulus is a "material constant". One possible motivation of this perhaps comes from the classical Freudenthal's interpretation of Weibull modulus depending on the crack size distribution, which however assumes the cracks to be at large distance one from the other. It is here found with simple numerical experiments with collinear cracks that Weibull distributions tend to be obtained also with interaction taken into account, but the Weibull modulus depends on both the crack size distribution and the distribution of ligaments. Hence, Weibull modulus should not be considered a "material constant" or to correspond to an "intrinsic" microstructure of the material, as assumed in many industrial applications and commercial postprocessors of FEM softwares, even in the case of a varying stress fields. In the limit case of a crack or sharp notch this leads to paradoxically a zero scale parameter (and the usual Weibull modulus). Hence, in the case of a blunt notch, we suggest the Weibull modulus would vary depending on the distribution of cracks, their distances, and the interaction with the geometry and stress field. Only numerical simulations where the distribution of cracks is directly included in the geometry under consideration can provide the correct scale factor and Weibull modulus.

Keywords: Size effect; Weibull theory; Brittle fracture

1. Introduction

The strength of brittle materials and size effects is a classical subject which has attracted the interest of many of the greatest scientists of all times including Galileo (1638), who first recognized the "size effect" as the departure from failure at the nominal failure stress, Mariotte (1686), and others (see reviews by Bazant (2002, 2004)). This problem was then studied experimentally by Weibull and resulted in his well-known distribution (Weibull, 1939, 1951), although in 1928 this distribution had been already derived by Fisher and Tippett (1928). Since the tail of similar results is a power-law, Weibull suggested that statistical size effect is also a power-law with the exponent being the coefficient of variation of material strength. Weibull theory

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was amply confirmed and perhaps even used more widely or less safely than the scope and the original motivation would suggest. First of all, it is recognized that a large class of materials have cracks which are in the transition from brittle behaviour to ductile behaviour (it is possible to define a fracture process zone: only when the latter is negligible compared with the size of the structure, the material is perfectly brittle and follows linear elastic fracture mechanics, LEFM). More recently, fractals are used to interpret size effects perhaps more efficiently (Carpinteri, 1994; Mandelbrot et al., 1984), recognizing the multiscale nature of power-law distributions can be more advantageously discussed within this context than in the context of statistics, even if not all authors agree on this advantage of fractals (see Bazant and Yavari, 2005).

One unjustified assumption often made in the engineering community and literature is to assume that the size effect power-law exponent, i.e. the coefficient of variation of material strength, is a material constant. Is this the case? This paper will illustrate why this is not, with a simple example of collinear cracks. This is by no means of interest of realistic materials (why would the crack be aligned only along a line), but to provide a simple way to define their distance and study the effect of interaction.

But where does the assumption of Weibull modulus as a material constant come from? Is there any reason for this? One possible explanation is that most people consider the well-known interpretation of Weibull theory by means of the model of non-interacting cracks, hence attribute Weibull modulus to the crack size distribution, and translate this as a material properties into any other condition, even including different geometrical and loading conditions (this latter aspect will be discussed towards the end of the paper). Freudenthal (1968) indeed showed that a Weibull distribution is obtained with an inverse power-law distribution of *flaws*, but neglecting mutual interaction in the stress fields surrounding each flaw, and applying Griffith equation which specifies the critical length a of an elliptical crack in terms of the stress intensity σ

$$\sigma\sqrt{a} = k = \text{const} \tag{1}$$

where the constant k depends on the Young's modulus E, Poisson's ratio v and the rate of work Γ_c to produce a unit area of crack.

In particular, according to *Weakest Link Theory* (WLT), the fracture of a specimen is identified with the unstable propagation of the most "critical" crack (the largest in a uniform stress field). Peirce (1926) was the first to formulate WLT and recognized the close relation of this model to the theory of extremes values, stating that the distribution of smallest values tends in the limit of large number of samples to be one of the *asymptotic distributions*, regardless of the initial population (Gumbel, 1958). Only two kinds of physically significant distribution functions of extreme values exist: one function represents the extremes of unlimited initial populations, described by functions that converge towards zero for $|x| \to \infty$ at least as fast as the exponential function $\exp(-x)$; the other function represents the extremes of initial populations (called Cauchy type distributions) that are limited for $x = 0$ and converge towards zero for $|x| \to \infty$ as fast as an inverse power-law x^{-m}. For example, for a Cauchy type distributions of crack size a, the function distribution of the largest cracks was derived by Frechet (1927)

$$F_a(a) = \exp\left[-\left(\frac{a}{u}\right)^{-\alpha}\right] \tag{2}$$

with u characteristic size. By using Griffith's equation, Eq. (1), this distribution can be converted into a distribution of strength

$$F_\sigma(\sigma) = 1 - \exp\left[-\left(\frac{k^2}{\sigma^2 u}\right)^{-\alpha}\right] = 1 - \exp\left[-\left(\frac{\sigma}{\sigma_0}\right)^{2\alpha}\right] \tag{3}$$

with $\sigma_0 = k/\sqrt{u}$ and such distribution coincides with the two-parameter Weibull distribution with modulus $m = 2\alpha$.

This model can explain experimental strength data of brittle material to some extent, but when there is interaction between cracks, or between cracks and the gradient of the stress field, the modulus is found to depend significantly on loading and geometrical factors, and is not a material constant, as noticed already experimentally by various authors (Milella and Bonora, 2000).

In order to attack the problem of interaction, we start from a simple case, generating distributions of collinear cracks in a large plate under uniform tension, and finding the strength distributions by an efficient dual boundary element method (DBEM) (Portela et al., 1993; Aliabadi, 2002).

2. Formulation

The geometry of the problem is shown in Fig. 1 where an infinite plate with N collinear cracks is loaded by uniform remote tension σ_∞. The model is similar to that considered by Zhang et al. (1998).

In particular, two cases was considered:

- N collinear cracks with arbitrary lengths and equal ligament sizes;
- N collinear cracks with arbitrary ligament sizes and equal lengths.

The ultimate stress σ_r has been evaluated as

$$\frac{\sigma_r}{\sigma_\infty} = \frac{K_{\mathrm{IC}}}{K_{\mathrm{I}}} \tag{4}$$

where K_{I} and K_{IC} are, respectively, the highest stress intensity factor in a specimen and the critical stress intensity factor of the material.

To define the size of the samples, i.e. the number of specimen in each sample, the following equation (proposed by Smart et al. (2003) and relating the Weibull modulus m to the number of experimental data) can be used:

$$\frac{m_{\mathrm{sd}}}{m_0} = \frac{1}{\sqrt{N_g}} \tag{5}$$

where N_g is the number of data and m_0 and m_{sd} are the expected value and standard deviation of the Weibull modulus. Eq. (5) shows as the scatter of the modulus m decreases with the number of data N_g and the larger is m the larger has to be N_g.

Fig. 2 show the values of Weibull modulus obtained by a two parameters regression on samples with N_g from 25 to 500. For each specimen, the ligament size is distributed with *inverse power-law* ($r = 5$ and $d = 2$), i.e. we suppose that the crack length is a random variable distributed with a probability density function $p(a) = 1/a^r$.

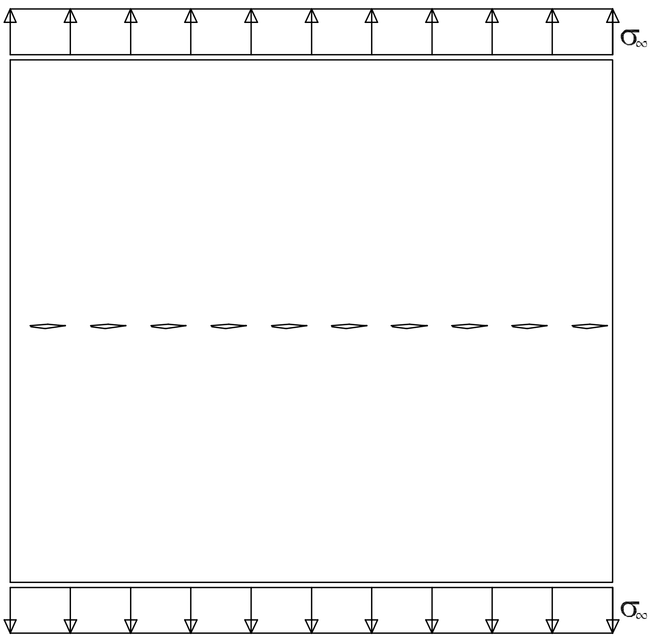

Fig. 1. Infinite plate with N collinear cracks, loaded on the edges by uniform remote tension σ_∞.

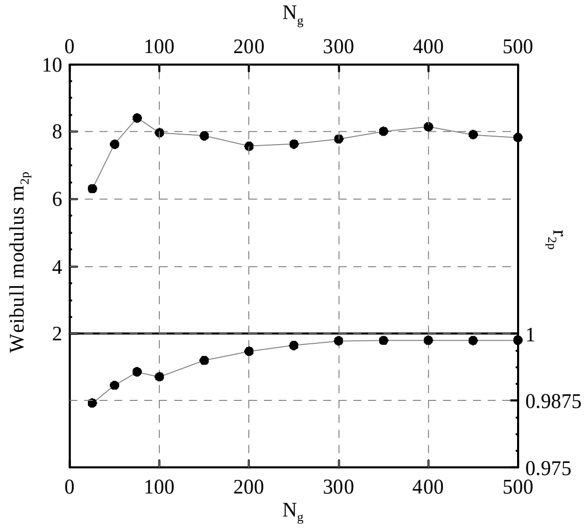

Fig. 2. Variation of the Weibull modulus with the number of specimens N_g (two parameters distribution).

The number N_g of specimens needed to identify with enough reliability the correct distribution can be very large. However, we set $N_g = 100$ as the estimated error is around 2%. Further, the number of cracks of the specimens is bounded to $N = 50$, finding a trade-off between reliability and computational costs.

3. Cracks with length 2*a* distributed with *inverse power-law* and equal ligament size *d*

In this section a first investigation on the interaction effects between the cracks is shown. We considered 50 collinear cracks with identical ligament size d. A distribution with *inverse power-law* is considered for the cracks length $2a_i$. Twelve samples, each with $N_g = 100$ specimen, have been generated with the following values of ligament size d and exponent of power-law r:

$$d \in [0.002; 0.02; 2]$$
$$r \in [2; 3; 5]$$

The maximum stress intensity factor $K_{\mathrm{I,max}}$ has been evaluated and by (4) the ultimate stress σ_r (each elasticity problem being solved by an efficient dual boundary element formulation). In Fig. 3 the failure probability is shown vs the normalized strength (σ_r/σ_∞) for $r = 2$ and $d = 0.2$. The numerical results are better fitted by the three parameters Weibull distribution than two parameters one (the correlation coefficient is 0.993 for the three parameters Weibull distribution, 0.992 for two one). However, the difference is very small and, for this reason, the simpler two parameters distribution is used in the following.

Fig. 4 shows the variation of the Weibull modulus m_{2p} of the ultimate stress distribution with the exponent r of the cracks' inverse power-law. The curves are obtained for different values of the ligament d. Notice that the Weibull modulus increases upon increase of the exponent r, stating that the scatter of the material strength decreases. In particular, for $d > 0.002$ the numerical curves are very close to theoretical one (dashed dot line) which is obtained by assuming no interaction between the cracks ($m_{2p} = 2(r - 1)$). Only when the ligament is very small ($d = 0.002$) the interaction between the stress fields around the cracks becomes significant and the Weibull modulus deviates from the theoretical one. In fact, in this case the ratio between the ligament and the crack medium size (a_{med}) is always less than one, as indicated in the table shown in Fig. 4. As expected, interaction effects are significant only when cracks and their ligaments are of comparable size.

This can be interpreted as a transition zone between the applicability field of the Weibull theory (in which maximum scatter of the results is expected) and that of the deterministic results of the constant size ligaments.

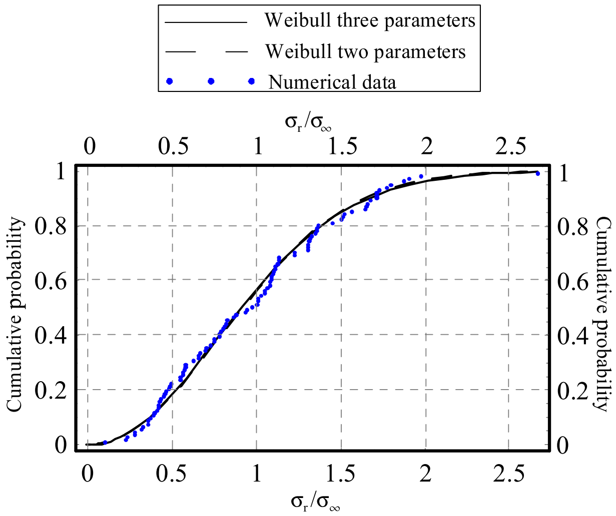

Fig. 3. Probability failure vs normalized strength ($r = 2$ and $d = 0.2$).

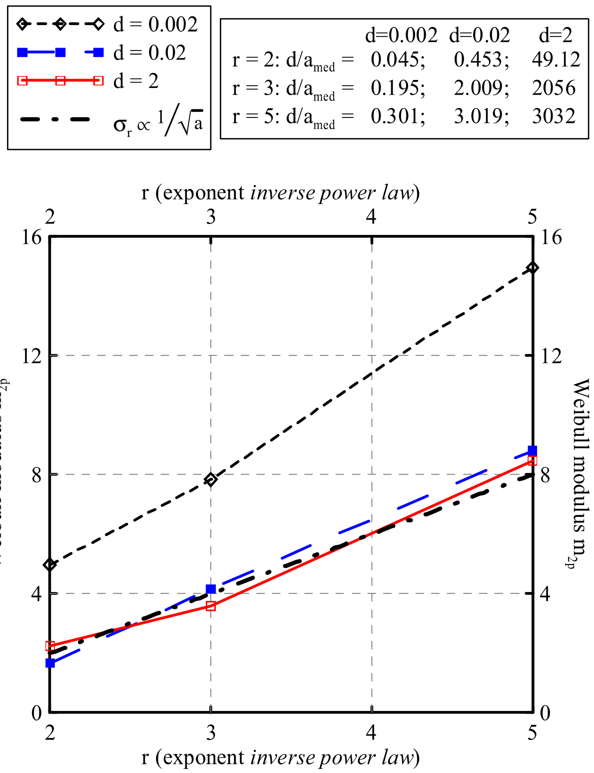

Fig. 4. Variation of the Weibull modulus m_{2p} with the exponent r of the inverse power-law, for different values of ligament d.

If the ligaments were to be distributed, their statistics would imply in turn a different Weibull exponent in the resulting strength distribution.

In Fig. 5 the variation of the scale parameter σ_0 with the exponent r of inverse power-law, for different values of ligament d is plotted. The interaction between the cracks involves a larger concentration of the stress fields around the tip (larger K_I) and a nominal strength reduction of the specimen. In fact, the scale parameter

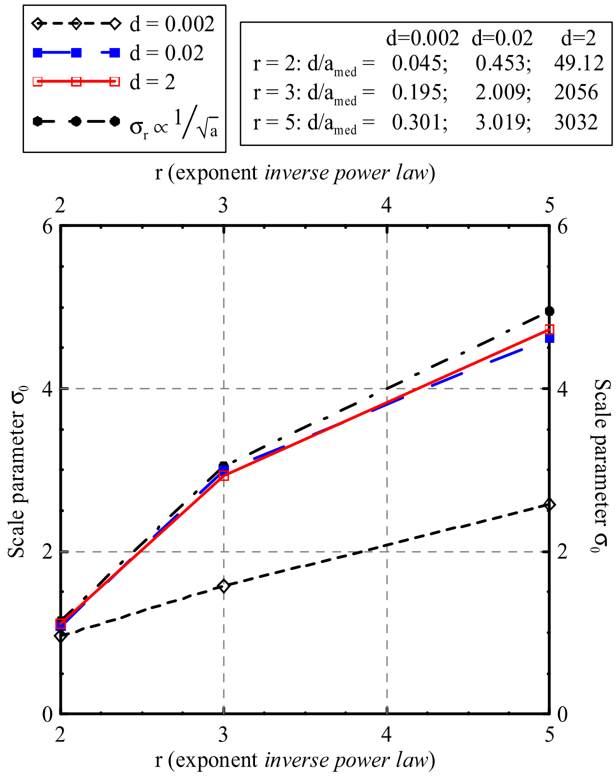

Fig. 5. Variation of the scale parameter σ_0 with the exponent r of the inverse power-law, for different values of ligament d.

σ_0 decreases by decreasing the ligaments size d, and moves towards an upper bound represented by the theoretical scale parameter (dashed dot line) which is evaluated in according to the weakest link hypothesis, i.e. by considering only the largest crack in its isolation.

An example is shown in Fig. 6, where a comparison between theoretical and numerical results (and the respective Weibull regression curves) is presented for ligament $d = 0.002$ and exponent $r = 5$. The effective nominal strength is lower than theoretical one.

Therefore, for cracks with length $2a$ distributed with *inverse power-law* and equal ligament size d, the Weibull modulus significantly deviates from the theoretical one only for small values of the ratio between the ligament d and the crack medium size a_{med} ($d/a_{med} < 0.5$).

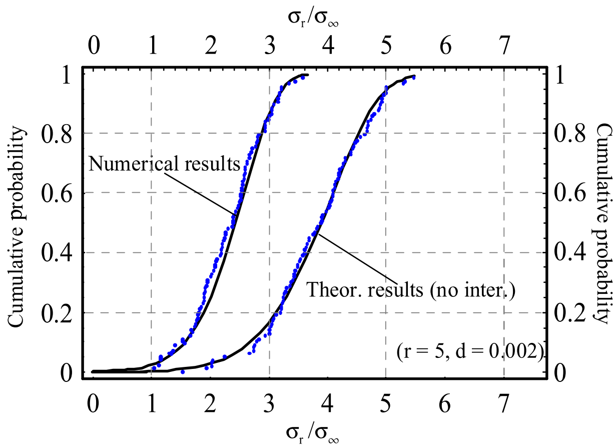

Fig. 6. Comparison between failure probability of the theoretical case (without interaction) and numerical one ($r = 5$, $d = 0.002$).

4. Cracks with constant length 2*a* and ligament *d* distributed with *inverse power-law*

In this section, the case of 50 cracks with constant length $2a$ and ligament d distributed with probability density function $p(a) = 1/a^r$ is studied. Nine samples, each with $N_g = 100$ specimens, have been generated with the following values of crack length $2a$ and exponent of power-law r:

$$2a \in [0.02; 2; 20]$$
$$r \in [5; 8; 10]$$

Similarly to the above section, we shall find a distribution of strength. However, differently from the previous case, here we need to reason in terms of load, because the Griffith equation is now modified for external cracks (Fig. 8)

$$P_r = \text{const}\sqrt{d} \tag{6}$$

where d is the ligament size. Hence, the cumulative probability function of the ultimate load P_r (with $P = \sigma \cdot L$, where L is the plate width size) is evaluated by

$$\frac{P_r}{P_\infty} = \frac{K_{IC}}{K_I} \tag{7}$$

Fig. 7 shows the variation of the Weibull modulus m_{2p} of the distribution P_r/P_∞ with the exponent r of the inverse power-law, for different values of cracks length $2a$. The Weibull modulus shows a similar trend to that of the specular case (discussed in the above section): an increase of exponent r yields an increase of modulus m_{2p}; vice versa an increase of the crack size $2a$ yields a decrease of m_{2p}. The theoretical curve of the Weibull modulus is plotted with dashed dot line. It is deduced by Eq. (3) assuming for the ultimate load P_r Eq. (6).

In fact, Fig. 7 shows as increasing the crack size $2a$, i.e. approaching the limit case of external crack of Fig. 8, the numerical curves move towards the theoretical one. Hence, in the limit of very large cracks with

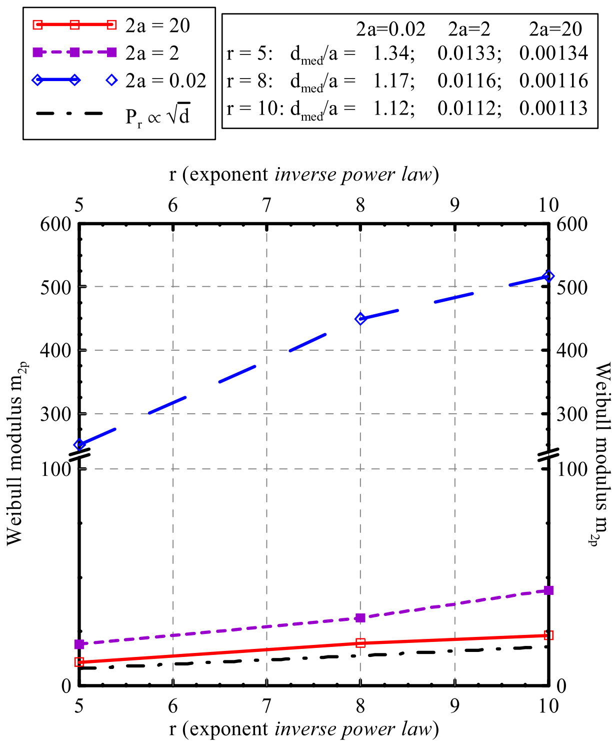

Fig. 7. Variation of the Weibull modulus m_{2p} with the exponent r of the inverse power-law, for different values of cracks length $2a$.

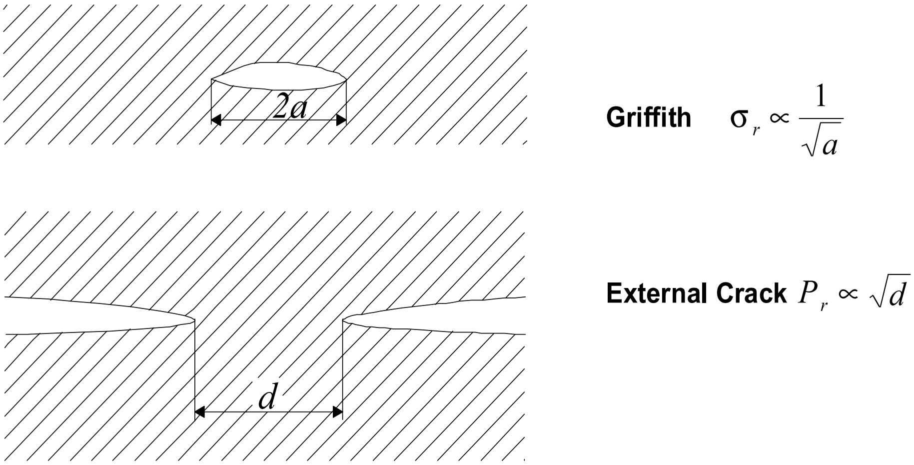

Fig. 8. Isolated and external crack.

respect to the ligament size (in our numerical experiments for $d_{\rm med}/a \simeq 10^{-3}$), we expect the applicability of a specular theory to that of Freudenthal.

Also, the increase of the Weibull modulus (i.e. the decrease of the scatter) when the crack length reduces with respect to ligaments size denotes that a transition exists between the statistical distribution in the sense of Weibull (in which it is necessary to reason in terms of probability of fracture) and the determinism of the LEFM of the single defect (in which deterministic results are obtained for the ultimate strength because the crack size is constant). If the crack sizes were to be distributed, the transition would be between two statistics.

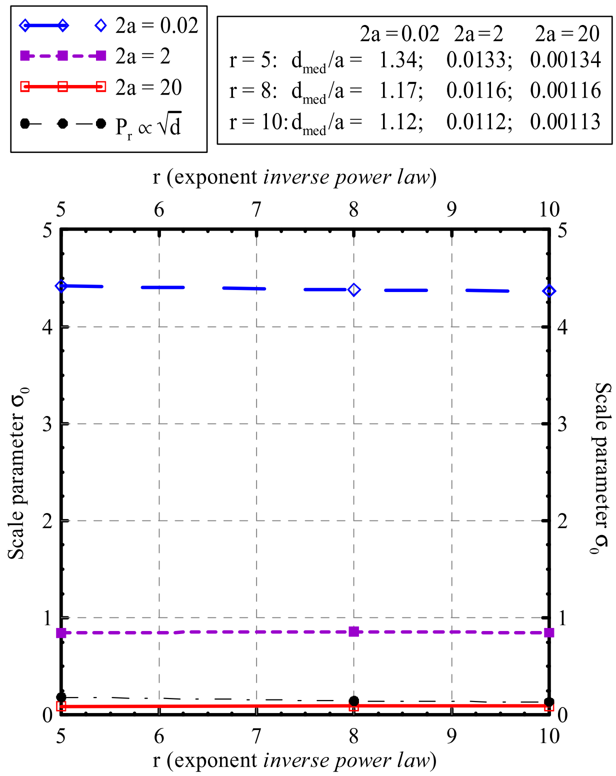

Fig. 9. Variation of the scale parameter σ_0 with the exponent r of the inverse power-law, for different values of cracks length $2a$.

The specular theory is confirmed in Fig. 9 where the variation of the scale parameter σ_0 with the exponent r of the inverse power-law is plotted for different values of cracks length $2a$. However, notice that, contrary to the case discussed in Section 3, the theoretical scale parameter, evaluated in according to Eqs. (3) and (6), can be interpreted as a lower bound.

5. Discussion: Weibull paradox

The use of Weibull's modulus as a material constant, and the idealization of a given set of tests into a Weibull "material" leads to further extensions of Weibull's theory and of the Weakest Link Theory to the Weibull integral, often used even in well-known softwares like CARES by NASA, where the probability of failure of a given specimen is assessed by multiplying the "local" probability of failure for each infinitesimal element. This process carries strong assumptions, since essentially an entire distribution of cracks which in some experiment has lead to a given Weibull modulus is concentrated in infinitesimal elements of material, and the specimen's geometry can no longer affect the Weibull modulus. A limit case is when the specimen is notched or even deterministically cracked, in which the theoretical Weibull distribution found by integration would lead to the same Weibull modulus but zero scale parameter (because of the singularity in the stress field).

In fact, considering also the effect of the volume, in terms of probability of survival $R(V) = 1 - F(V)$, Eq. (3) can be rewritten as

$$R(\sigma, V) = \exp\left[-\left(\frac{\sigma}{\sigma_0}\right)^m \frac{V}{V_0}\right] = \exp\left[-n_c(\sigma)V\right] \tag{8}$$

The function $n_c(\sigma) = \frac{1}{V_0}\left(\frac{\sigma}{\sigma_0}\right)^m$ is a stress dependent "risk of rupture", where V_0 is representative of the material volume, σ_0 is a scale parameter and m is the Weibull modulus. The volume component is easily derived as it is the only one satisfying the WLT condition (see Freudenthal, 1968)

$$R(\sigma, nV) = [R(\sigma, V)]^n \tag{9}$$

Notice that this condition could be satisfied with $n_c(\sigma)$ being *any* function of the stress σ. The Weibull distribution simply corresponds to the case

$$n_c(\sigma) = \left(\frac{\sigma}{\sigma_0}\right)^m \quad \text{or} \quad n_c(\sigma) = \left(\frac{\sigma - \sigma_u}{\sigma_0 - \sigma_u}\right)^m \tag{10}$$

where the latter case is for three parameter distribution. For non-uniform stress field, we immediately derive the product as an integral,

$$R = \exp\left[-\frac{1}{V_0}\int_V \left(\frac{\sigma}{\sigma_0}\right)^m \mathrm{d}V\right] \tag{11}$$

However, writing WLT in this form is purely speculative as it corresponds to effectively assuming infinitesimal volumes of material non-interacting with each other—a distribution of cracks cannot be collapsed to a single material point! Therefore, the larger the stress gradient, or the larger the cracks with respect to their distance, the larger the deviation we expect from the simple Weibull case.

In order to see the effect of gradient, let us consider the case of a macroscopic crack or sharp notch. Using the Williams' asymptotic stress field for a notch $-\alpha < \theta < \alpha$,

$$\sigma(r, \theta) = Ar^{-p} g(\theta) \tag{12}$$

where A is an arbitrary constant, p is a dimensionless exponent and g is some function, we obtain

$$R = \exp\left[-\frac{1}{V_0}\int_{-\alpha}^{\alpha}\left(\frac{Ag(\theta)}{\sigma_0}\right)^m \mathrm{d}\theta \int_{\rho_1}^{\rho_2} r^{1-mp}\,\mathrm{d}r\right] \tag{13}$$

Now,

$$\int_{\rho_1}^{\rho_2} r^{1-mp}\,\mathrm{d}r = \frac{\rho_2^{2-mp} - \rho_1^{2-mp}}{(2-mp)} \tag{14}$$

There are two possibilities. If $mp > 2$, the integral will be unbounded when $\rho_1 \to 0$, whereas if $mp < 2$, it will be unbounded when $\rho_2 \to \infty$. For $mp = 2$ it is unbounded at both limits. Thus, for all values of m, p the probability of survival is zero, but for different reasons. For $mp > 2$, failure occurs because of the high stress in the notch, whereas for $mp < 2$ it occurs because of the unbounded volume of lightly stressed material away from the notch. In the latter case, bounded results would be obtained by considering the actual stress field in the finite body.

However, the main conclusion is that the mean value of strength is likely to be zero, whereas we expect it to be finite (as after all the macroscopic crack is not qualitatively different from the distribution of cracks giving the Weibull statistics). Also, there is no particular reason to expect the scatter to be given by the Weibull modulus in the case of pure tension. In fact, Weibull modulus depends on loading and geometrical factors, and is not a material constant as noticed already experimentally by various authors (Milella and Bonora, 2000). A Monte Carlo simulation for this case, conducted similarly to what done in the present paper, would certainly lead to a different Weibull modulus (intermediate between that of the material and the infinite modulus resulting from the deterministic crack) and a finite scale parameter. One obvious but quite empirical solution is to remove from integration a certain region ahead of the crack or notch tip, but this does not recognize the real nature of the problem.

6. Conclusions

The Weibull distribution is very commonly found in experiments, and we have confirmed it generally applies even if interaction between defects is considered. However, we also have shown it does not necessarily correspond to a "material property" modulus, and a geometry and loading condition dependent scale parameter. Indeed, the Weibull modulus can vary because of interaction between the cracks or between the cracks and the stress field. Hence, it is perhaps over-simplifying to assume Weibull as a material constant.

We have defined a specular theory for distributed ligaments and cracks very much larger than the ligaments, and in this case, not the crack distribution, but the ligament distribution gives the Weibull modulus. When both the cracks length and the ligaments sizes are statistically distributed the expected Weibull modulus will be a weighted average between the modulus of the cracks distribution and that of the ligaments distribution, with weight depending on the ratio between the characteristic sizes of cracks and ligaments.

Finally, we have demonstrated that, in the limit case of specimen notched or even deterministically cracked, the use of the classical Weibull "theory" by integration leads to a paradox. This paradox would be removed in a direct simulation, where different Weibull modulus would be found and a finite scale parameter, both depending on the interaction between the distribution of cracks and the stress field. This suggests that some care is needed in using the classical Weibull "theory" when there is possibility of interaction of defects, or interaction of defects with the stress field gradient.

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